Fluid Statics (Hydrostatics):

Hydrostatics deals with the study of fluids at rest with a focus on the forces acting on the surroundings. The pressure P defined as the force exerted on a unit area of fluid has units in N/m^2 or bars (10^5 N/m^2), Pressure in the fluid at rest increases with increase in depth of fluid.

For fluid at rest, all forces acting on a body of fluid are in equilibrium i.e. action and reaction forces are perpendicular to boundary surfaces. For fluid in motion, force within body of fluid exhibit a shear component therefore forces are not all perpendicular to boundary surfaces. The only force supporting a column of fluid at rest is force acting upwards as shown in figure 1.

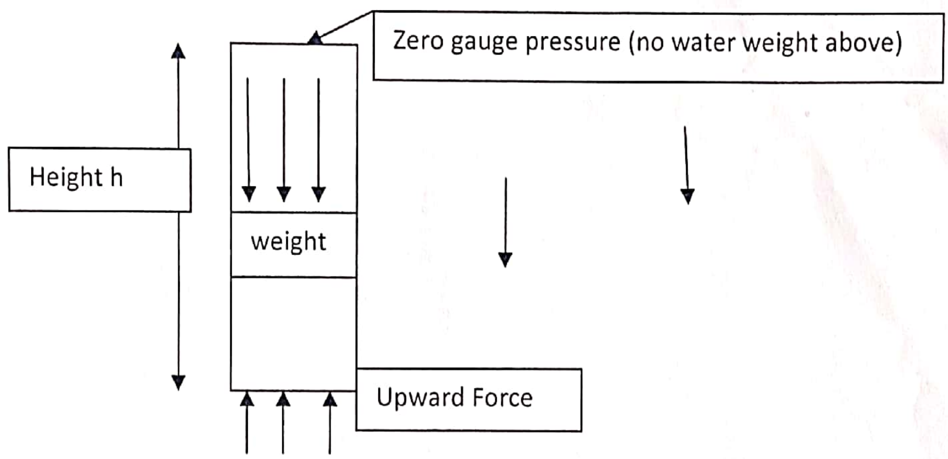

Figure 1: Column of Water with depth h.

Weight of column of water (acting downwards) must be equal to the upward force:

Specific Weight of water = ρg

Therefore the weight of the column of water is specific weight of water x volume = ρgAh

Where ρ is the density of water

g is the acceleration due to gravity

A is the horizontal cross sectional area of the column of water

and h is the depth of the column of water.

Upward force = pressure P x Area A = PA

Therefore PA = ρgAh

Pressure P = ρgh which is called the basic hydrostatic equation. P is the gauge pressure, this is the pressure with respect to the Atmospheric pressure P_{Atm}

Therefore, Absolute Pressure $P_{ABS} = P_{Atm} + \rho gh$

Negative gauge pressure is pressure below atmospheric pressure, which is called vacuum pressure.

Gauge pressure is zero at atmospheric pressure.

When using gauge pressure, atmospheric pressure is taken as datum.

The equation $P = \rho gh$ can be re-arranged to

$h = P/\rho g$ which is called the pressure head.

Points of Equal Pressure in Hydrostatic Fluid:

For hydrostatic fluid with a free water surface, the hydrostatic pressure varies with increase in depth; all points along a horizontal surface of fluid at rest are subjected to the same hydrostatic pressure. It must however be noted that for equal pressure to exist along a horizontal surface of fluids at rest:

1. The points considered in the surface must lie along/ in the same liquid.
2. The points must be continuous along the same elevation.
3. Continuity must exist within the fluid under consideration.

Measurement of Pressure:

Piezometer:

If a body of liquid (sealed) under pressure is pierced with a vertical transparent tube (called a piezometer), the liquid in the tube will rise to a height h defined as:

$h_w = P/\rho g$ (pressure head).

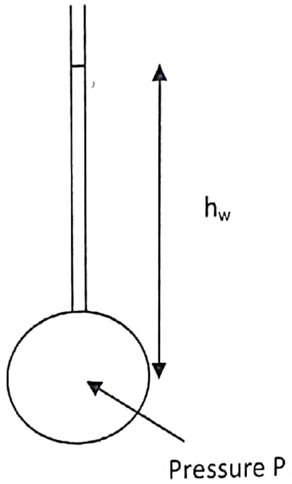

Figure 2: Piezometer

The use of the Piezometer is limited as the height of the water column in the tube becomes unreasonably large for large pressure measurements. The use of the Piezometer in measuring pressures in gas pipeline is un-practicable. A range of Pressure measuring devices has therefore been developed.

Manometer:

The manometer is a U – shaped transparent tube used in measuring pressures. The fluid in the manometer is usually different from the fluid of which the pressure is to be measured (usually water). This gauge fluid has a different density from water; mercury is usually used as this fluid is immiscible with water and does not reach chemically.

Fig. 3: The Manometer

If the gauge pressure in the body of fluid to be measured is zero, the surface of the gauge fluid in the left hand of the manometer will be at the same horizontal level with the open surface of the manometer fluid at the right hand of the manometer.

Example 1: Show that the pressure head in a body of fluid (water) with pressure P measured using a mercury manometer is given by:

$$P/(\rho_w g) = [(\rho_m/\rho_w) h_2] - h_l$$

Where P = Pressure in the body of fluid

ρ_w = density of water

g = acceleration due to gravity

ρ_m = density of mercury

h_2 = height of column of mercury on right limb above datum x-x (where x-x is the horizontal level on the left limb of manometer at the interface between the water and mercury)

h_l = height of water column in the left limb of manometer.

Solution:

A horizontal datum x-x is set at the interface of the water and mercury level,

The total pressure on the left side of the manometer must be equal to the total pressure on the right side of the manometer from the set datum.

LHS of manometer: Total Pressure at LHS of manometer is given by Pressure at the centre of body of water plus pressure due to the height h_l of water in left limb:

$$P_{x\text{-}x} = P + \rho_w g h_l \quad \ldots\ldots\ldots\ldots(1)$$

RHS of manometer: Total Pressure at RHS of manometer is given by height of mercury from the datum x-x (since top is opened to atmosphere, atmospheric pressure is ignored to give gauge pressure):

$$P_{x\text{-}x} = \rho_m g h_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

(For a continuous homogeneous fluid, pressure is constant along any horizontal datum, therefore pressure at datum on LHS of manometer is equal to pressure on RHS of manometer)

Equating (1) to (2),

$P + \rho_w g h_1 = \rho_m g h_2$

$P = \rho_m g h_2 - \rho_w g h_1$ ....(3)

$P/\rho_w g = [(\rho_m/\rho_w)h_2] - h_1$

Other pressure measuring devices:

The sloping manometer: The manometer is connected directly to the body of fluid in a container with a much larger cross section than the manometer tube via a sloping transparent tube. The angle of slope is given as Θ.

The Bourdon gauge: This is made by a tube the shape of a question mark with a sealed upper end which is connected to a pointer. As the pressure in the tube increases, the deformation in the tube causes the pointer to move through a corresponding angle.

Example 2:

Comment:

(a) on the height; h_w of the liquid in a pizometer (fig. 2) used to measure the gauge pressure in a pipe which contains water at a pressure of $50KN/m^2$.

(b) on the height h_m of the gauge fluid (mercury) of a manometer (fig. 3) if inserted instead of the piezometer, also used to measure the gauge pressure in the same pipe which contains water at a pressure of $50KN/m^2$.

(Density of water is $1,000 kg/m^3$ and density of mercury is $13,600\ kg/m^3$.)

Solution:

(a) The piezometer reading given by the height $h_w = P/\rho g$

$h_w = 50 \times 10^3 \div (1000 \times 9.81)$

$h_w = 5.097m$

(b) The manometer reading given by the height of gauge fluid h_m is determined from equation (3) above (where $h_2 = h_m$ and $h_1 = y$):

$$P = \rho_m g h_m - \rho_w g y$$

Since deflection in the left hand limb of the manometer has to be calculated:

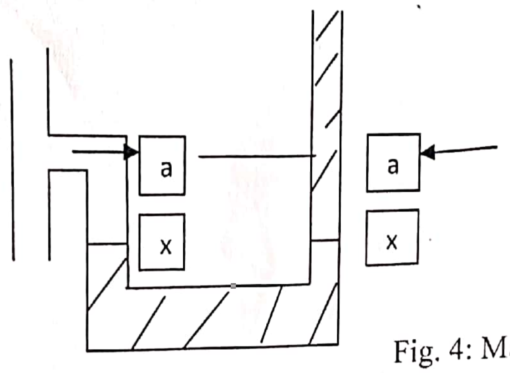

Fig. 4: Manometer reading with datum at x-x.

When gauge pressure in pipeline is zero, mercury level in U tube will be at level a-a.

When gauge pressure is increased in pipeline to $50KN/m^2$, left limb will deflect by a distance y and the right limb will move up by a distance y from a therefore total mercury distance on right hand limb will be 2y.

$$P_1 + \rho_w g y = \rho_m g h_m$$

$$50 \times 10^3 + 9.81 \times 1000\, y = 13600 \times 9.81 \times 2y$$

$$50 \times 10^3 + 9810y = 133416 \times 2y$$

$$50000 = 257022y$$

$$y = 0.1945$$

$$2y = 0.389m$$

h_m obtained using a mercury manometer is a much smaller value than y obtained using a piezometer.

Hydrostatic Pressure on Plane Surfaces immersed in a body of fluid:

Pressure forces in hydrostatic fluid are assumed to be perpendicular to the imaginary surfaces of the fluid. Action and reaction of forces in the fluid must be in equilibrium at all points.

Consider a flat surface with width B and length L suspended horizontally in a body of fluid at depth h.

$$P = \rho gh$$

Since the surface is horizontal, pressure is uniform across the whole surface.

Consider a plate vertically suspended in water as shown in figure 5 with the top edge at surface of the water:

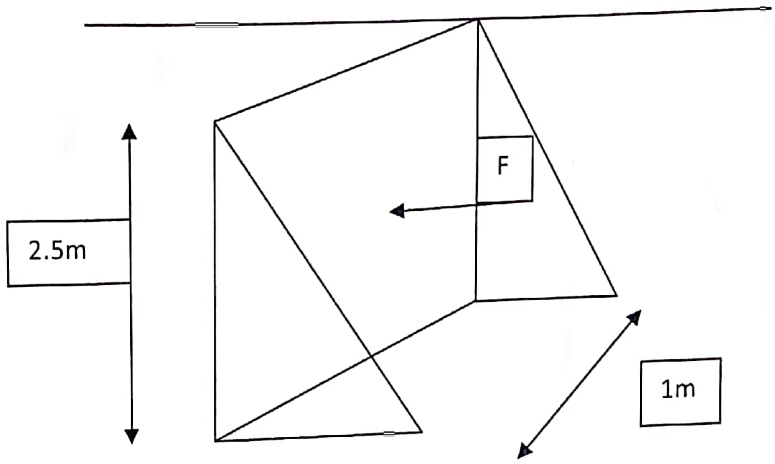

Figure 5: Plate vertically suspended in water.

Pressure P increases with depth therefore, Pressure is not constant with depth.

At the surface, gauge pressure is zero, at bottom, gauge pressure at depth 2.5m

$$P = \rho gh = 1000 \times 9.81 \times 2.5 = 24{,}525 N/m^2$$

Force on one side of the plate F = Pressure x Area

$$\text{Force } F = (24525 + 0) \div 2 \times 1 \times 2.5$$

$$= 30{,}656N$$

Locating the point of action of F depends on pressure P which is not constant throughout depth h and Area A (regular or irregular).

Consider a small elemental area of the plate δA at distance h below surface of the water:

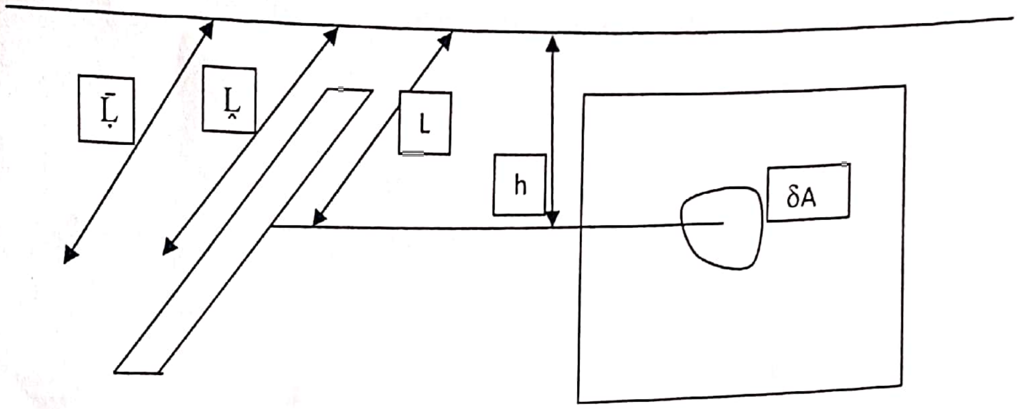

Figure 6: Elemental Area of Plate δA

$\delta F = P\ \delta A = \rho g h\ \delta A$

Total Force $F = \rho g \int h\ \delta A$ (ρg are constants)

But $h = L \sin \Theta$

Therefore $F = \rho g \sin \Theta \int L\ \delta A$

$\int L\ \delta A$ is geometrical characteristics of the shape and it is known as the First Moment of Area, given by $A\ \bar{L}$ where A is the Area of the plate and $\bar{L}$ is distance from origin to centroid of the plate.

Therefore: $F = \rho g \sin \Theta\ A\ \bar{L}$

To determine the location of the line of action of F, and angle for the elemental area:

δF produces a moment δFL about origin.

Moment $= \delta FL = \rho g h\ \delta A\ L$

$= \rho g\ L \sin \Theta\ \delta A\ L$

$= \rho g \sin \Theta\ L^2\ \delta A$

In the limit

$dFL = \rho g \sin \Theta\, L^2\, dA$

The quantity $\int L^2\, dA$ is the second moment of area with the symbol I

Moment about origin $= \rho g \sin \Theta\, I$

The distance from the origin to point of action of F

L_x = moment/ Force $= \rho g \sin \Theta\, I \div \rho g \sin \Theta\, A\, \bar{L}$

$= I/ A\bar{L}$

Where I from parallel axes theorem is given by

$I = I_o + A\, \bar{L}^2$

Where I is the second moment of area of a plane surface about origin O.

I_o is the second moment of area of the surface about an axis through its centroid.

(Note: if axis of plane passes through the centroid, then the second moment has a particular value I_o for any given shape, thus for a rectangle $I_o = BH^3/12$ where B = width and H= height and for a circular plane of radius R, $I_o = \pi R^4/4$. If the second moment of area about another axis is required e.g. x-x, then $I_{x\text{-}x} = I_o + A\bar{L}^2$ - this is the Parallel Axis Theorem)

Floatation and stability:

Buoyancy Forces:

Buoyancy forces are experienced in various Engineering designs including:

1. Gas lines laid in water logged areas
2. Rigs used offshore in oil exploration
3. Structural elements moved over water such as steel docks or lock gates

Consider a cylinder submerged in a body of fluid (water) with its axis vertical shown in Figure 1:

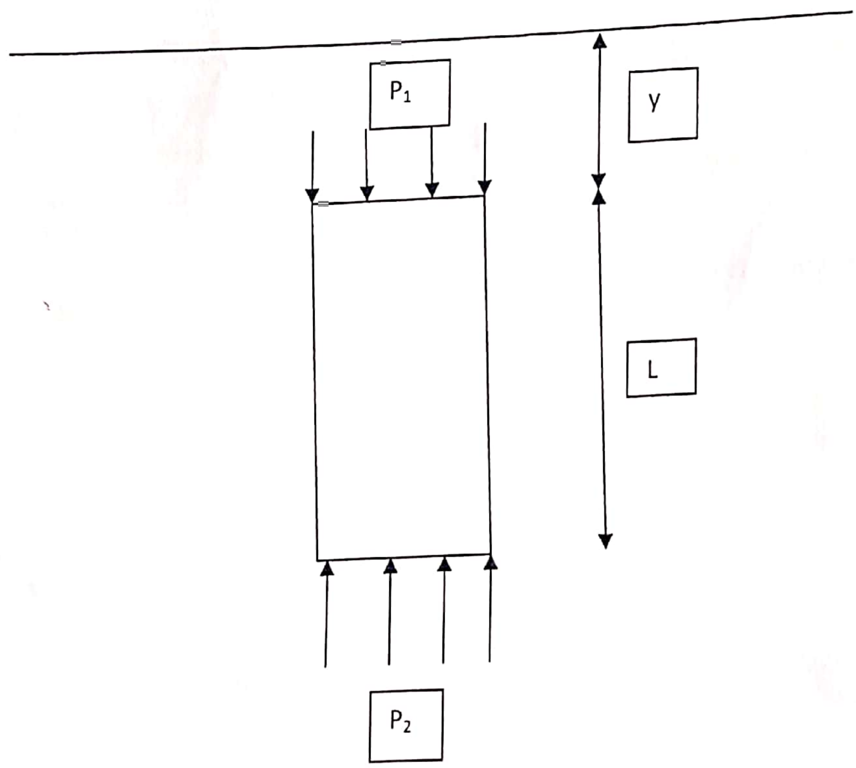

Figure 1: Pressure forces acting on immersed Cylinder

Vertical forces acting on the immersed body of horizontal cross sectional area A is due to pressure on the horizontal surface is given by:

$P_1A = \rho g y A$

The force acting at the horizontal surface at the bottom of cylinder is due to pressure P_2 equivalent to:

$P_2A = \rho g(y+L)A$

Total upthrust = $F_B = \rho g(y+L)A - \rho gyA$

Where LA is equal to the volume of the cylinder.

The weight of a floating body and the resultant pressure of the fluid on the surface area of the floating body affect the equilibrium between the forces acting on the body which also affects the stability of the floating bodies.

The Archimedes principle states that the upthrust (i.e. the upward vertical force due to the fluid) on a floating body immersed in a fluid is equal to the weight of the fluid displaced by the floating body.

The upthrust acts through the centre of gravity of the displac4d fluid which is also the centre of buoyancy B.

Conditions of Equilibrium:

Stable Equilibrium: A small displacement from the position of equilibrium produces a moment to restore the body back to the position of equilibrium.

Unstable Equilibrium: This is a condition in which small displacement results in an overturning moment tending to displace the floating body further from the position of equilibrium.

Neutral Equilibrium: The floating body remains at rest following a displacement on the body.

Stable Equilibrium: For bodies designed to float, the design must result in the body floating in a stable condition. For angular displacement on a floating body, a lateral displacement of the position of centre of Buoyancy B occurs (except for circular sections).

Consider a pontoon (a floating structure e.g. a dock) of a rectangular section shown in figure 2:

When the body is in equilibrium (i.e. upright floatation) the pressure distribution at the base of the body is uniform.

The centre of buoyancy lies on the same vertical line with the centre of gravity. The stability of the floating body is determined by the relative positions of the centre of gravity of the floating body G and the centre of buoyancy B (i.e. the centre of gravity of the liquid volume displaced by the floating body)

Since the pontoon is stable, the weight W of the body acts along the vertical centre line. The floating body is in equilibrium when the centre of gravity and its centre of buoyancy lie on the same vertical line.

A displacement on the body causes a rotation (or made to heel) about its longitudinal axis through an angle Θ as shown in Figure 3, the pressure distribution on the base of the body becomes non-uniform although linear.

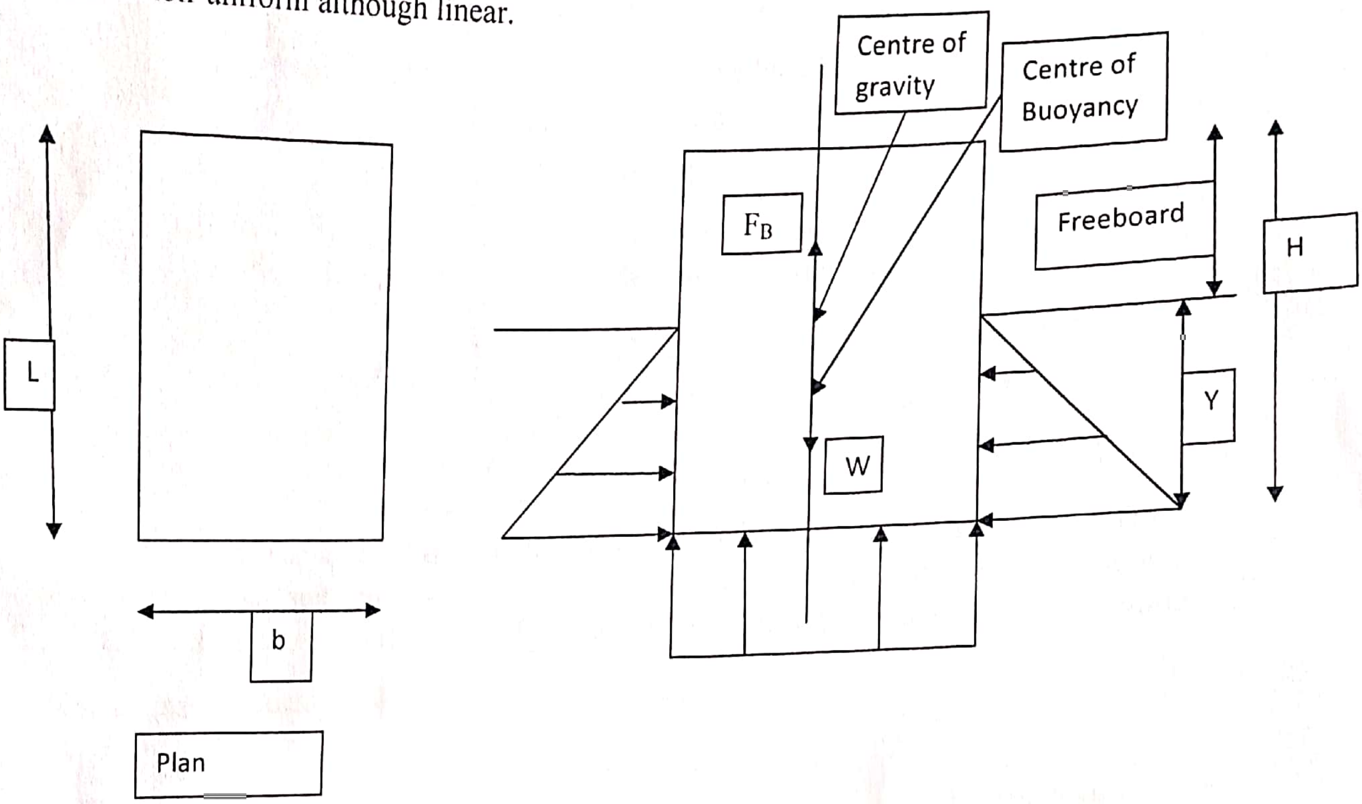

Figure 2: Plan and Section of Pontoon floating in water.

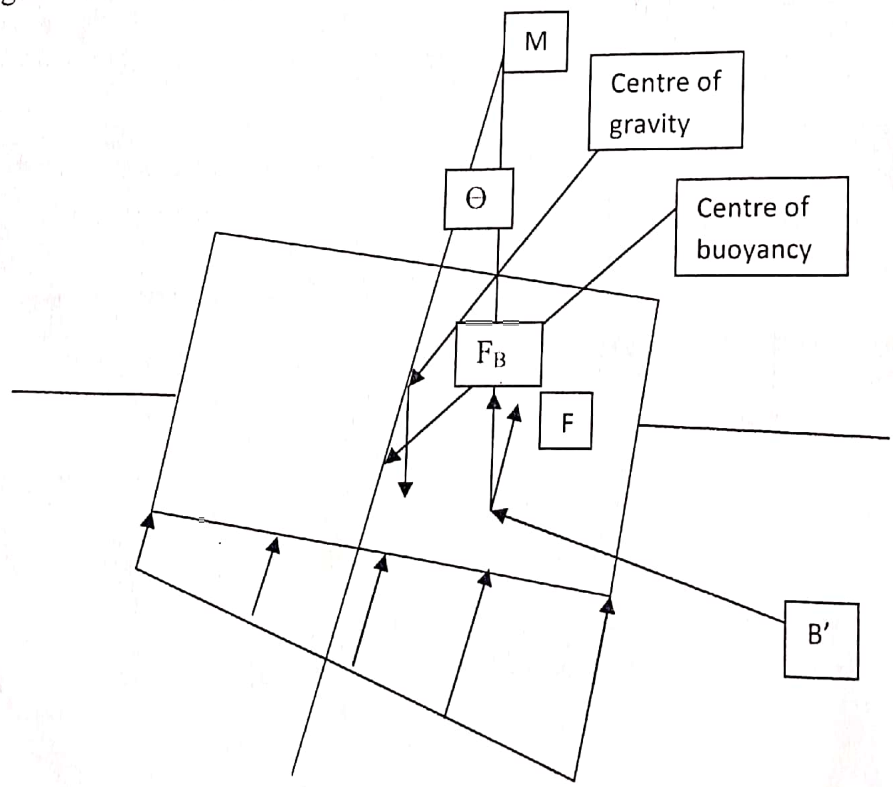

Figure 3: Pontoon displaced through angle Θ

F = Resultant Force on base of pontoon

M = Metacentre

F_B = Vertical component of Force F

B' = Point where Buoyancy force acts through

G = Centre of Gravity

B = Centre of Buoyancy

Consider a floating body with the shape in figure 4 which has been made to heel or list through an angle Θ. In this heeled position, the centre of gravity remains unchanged but the centre of buoyancy which is the centre of gravity of the new area is moved from B to B'.

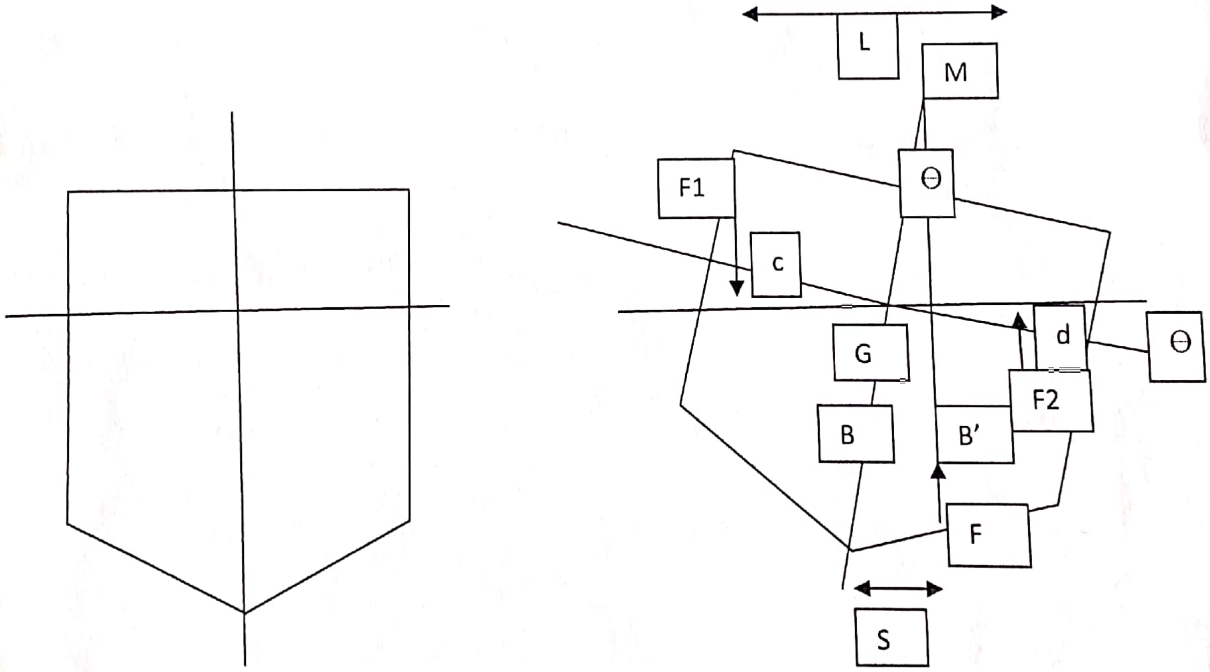

Figure 4: Floating object heeled through angle Θ

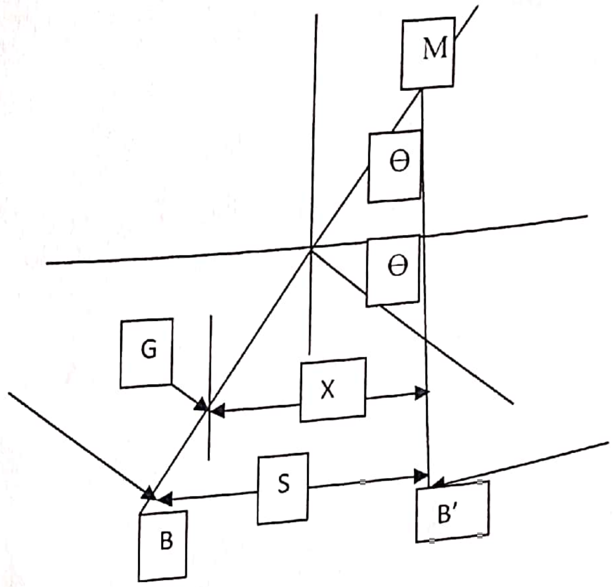

Figure 5: Enlarged angles showing $\boxed{\Theta}$

The buoyancy force F = ρg.Vol acting upward through B' and the weight of the floating body W acting downward through G will constitute a couple W multiplied by distance X, this couple will resist further over-turning and tends to restore the floating body to its original position. (Note: Vol = total volume of liquid displaced by body submerged).

The distance between the centre of gravity of floating body and the metacentre M is the metacentric height. The metacentric height gives the measure of the floatation stability of the body. When Θ, the angle of inclination is small, the position of the metacentre M does not result in an effective change with the tilting position.

The position of the metacentric height and the righting moment is determined as follows:

Heeling position of the floating body does not change the total weight of the body therefore the total volume of water displaced is also not changed. The tilt changes the shape of the displaced volume of water as shown in figure 4. In the heeled position, the total buoyancy force is shifted through a horizontal distance S, from B to B'.

The shape of the displaced volume of water is changed by removing prism c and adding prism d. The shift of buoyancy force from B to B' creates a couple F1 and F2 due to prisms c and d.

The moment of shifted buoyancy force F = ρg.Vol at B' about point B must be equal to the sum of the moments of the component forces F1 and F2:

ρg.Vol.S = moment of the forces F1 and F2

ρg.Vol.S = ρg.volume of prism. L

Distance S = (Volume of prism/ Vol).L ....(1)

Where Volume of prism = Volume of prism c or d and

Distance L is the horizontal distance between the centers of gravity of the prisms.

From geometry:

S = MB sin Θ ....(2)

Substituting (2) into (1):

MB = (Volume of prism/ Vol. sin Θ).L

For a small angle, sin Θ is approx equal to Θ, therefore:

MB = (Volume of prism/ Vol.Θ).L

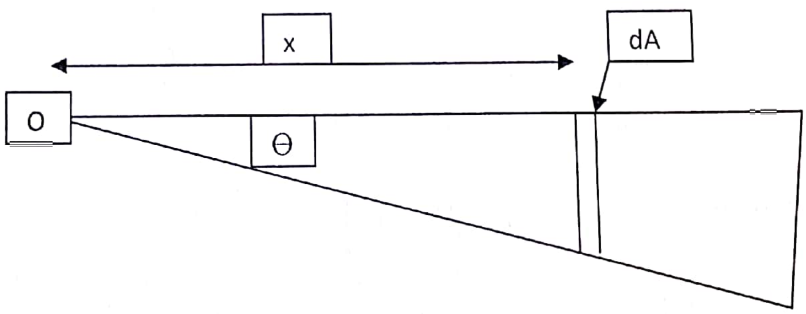

Figure 6: Enlarged prism d:

The buoyancy force produced by the prism d can be estimated.

Assume the horizontal area of a small part of prism is dA, located at a distance x from the axis of rotation O.

Height of prism is xtanΘ

For small angle Θ, xtan Θ can be approximated to xΘ.

Buoyancy force produced by small prism is ρgxΘdA

The moment of this force about the axis of rotation O is:

$\rho g x^2 \Theta dA$

Sum of all moments in the prism gives the moment of the immersed wedge. The moment produced by the force couple is:

ρg.volume of prism. $L = FL = \int_A \rho g x^2 \Theta dA = \rho g \Theta \int_A x^2 \, dA$

But $\int_A x^2 \, dA$ is the moment of inertia of the waterline, total cross sectional area of the floating body about the axis of rotation O.

$I_o = \int_A x^2 \, dA$

Therefore:

volume of prism. $L = I_o \Theta$

Therefore:

$MB = I_o / Vol$

The metacentric height which is the distance between the metacentre M and centre of gravity G is given by:

$GM = MB \pm GB = I_o / Vol \pm GB$

The sign ± shows the relative position of G with B. For greater stability, it is required that the centre of gravity be as low as possible. If G is lower than B this would be added distance to the metacentric height.

Righting moment for a tilt is given by:

$M = WGM \sin\Theta$

A floating body is therefore stable if the centre of gravity is below the metacenter, other the body is unstable.

A submerged body is stable if the centre of gravity is below the centre of buoyancy.

OR (Skip):

A shift from a uniform pressure distribution at the base of the pontoon when position is upright to a non uniform pressure distribution implies a shift in the position of the line of action of the resultant Force. Buoyancy force now acts through B'. A vertical line drawn through drawn through B' intercepts the centerline of the body at a point M called the metacentre.

The distance BB' can be found:

BB' = MB' sin Θ

Since MB' and BB' are unknowns, pressure distribution on the pontoon is used to evaluate the values.

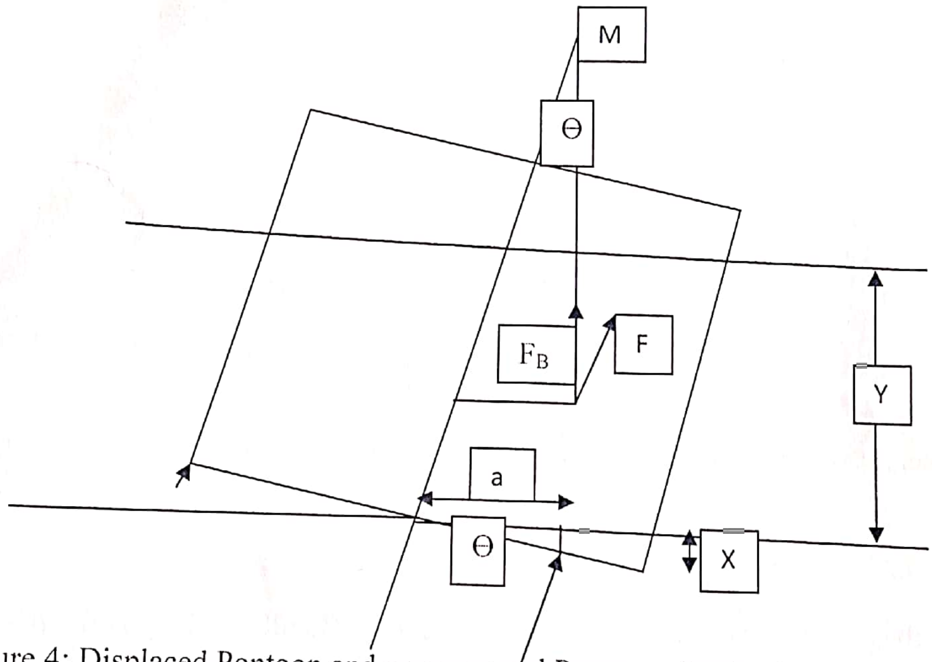

Figure 4: Displaced Pontoon and pressure and Pressure distribution

Pressure P acting at the base of displaced Pontoon at any distance a from the centerline is given by:

$P = \rho g(Y + h)$

Example:

A ship of total mass of 5,000,000kg floats on the sea in a stable condition. A portion of cargo, of mass 25,000kg shifts through a distance of 6m at right angles to the vertical plane of the longitudinal axis of the ship causing the ship to heel through an angle of 5 degrees. If the second moment of area I_o is given as 6,000m^4, calculate:

The Metacentric height

The distance between the: centre of gravity and centre of buoyancy for the ship.

(Density of Sea Water = 1025kg/m^3.)